DEPT. OF MATH./CMA UNIV. OF OSLO
PURE MATHEMATICS NO 2
ISSN 0806–2439 JANUARY 2011

STABILITY OF MERTON'S PORTFOLIO OPTIMIZATION PROBLEM FOR LÉVY MODELS

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Abstract. Merton's classical portfolio optimisation problem for an investor, who can trade in a risk-free bond and a stock, can be extended to the case where the driving noise of the log-returns is a pure jump process instead of a Brownian motion. Benth et al. [5], [6] solved the problem and found in the HARA-utility case the optimal control implicitly given by an integral equation. There are several ways to approximate a Levy process with infinite activity: by neglecting the small jumps or approximating them with a Brownian motion, as discussed in Asmussen and Rosinski [2]. In this setting, we study stability of the corresponding optimal investment problems. The optimal controls are solutions of integral equations, for which we study convergence. We are able to characterize the rate of convergence in terms of the variance of the small jumps. Additionally, we prove convergence of the corresponding wealth processes and indirect utilities (value functions).

1. Introduction

In Merton's [17] seminal paper on optimal portfolio management under uncertainty, it is proved that a risk averse investor will place a constant proportion of her total wealth in risky assets. Optimality is measured as the expected utility of terminal wealth, with a power or HARA utility function measuring the risk preferences of the investor. Moreover, the dynamics of the risky assets follow a geometric Brownian motion. The optimal proportion is given explicitly as the ratio of the excess return over the risk free, normalized by the volatility of the risky asset and the risk aversion of the investor.

The constant proportion rule is among the popular strategies for portfolio management in practice. Merton's portfolio selection problem has also gained a lot of attention in the scientific literature over the years, with generalizations in various directions. Recent extensions of the original Merton problem include the case of stochastic coefficients in Delong and Klüppelberg [10], and bounded downside risk through restrictions on Value-at-Risk and Expected Shortfall in Klüppelberg and Pergamenchtchikov [15]. For a general treatment and discussion, we refer to Øksendal and Sulem [20].

One stream in the literature relevant for our considerations focuses on analysing the effects of more realistic models for the risky asset price dynamics on the optimal portfolio management problem. For example, Benth, Karlsen and Reikvam [5] examined Merton's problem when the risky asset price dynamics is given by an exponential pure-jump Levy-process. Also in this case it is optimal to invest a constant proportion of the investor's wealth in the risky asset, however, the proportion is given in terms of a solution of an integral equation involving the excess return of the asset and the characteristics of the jump processes. One may easily include consumption into the portfolio problem, regaining qualitatively similar solutions as in the classical Merton's problem. Emmer and Klüppelberg [12] examine constraints of an upper bound for the risk when stock prices follow an exponential Lévy process. There is also work on optimal portfolios with HARA-utility in multidimensional cases, see for example Calleg and Vargiolu [8], where assets are driven by a multidimensional Poisson process, or Pasin and Vargiolu [21] in the case of exponential additive processes.

Date: January 26, 2011.
Key words and phrases. Stochastic control, portfolio optimization, Lévy processes, utility optimization, integral equations, approximations.

Lévy processes are popular in financial modelling since they are able to explain many of the stylized facts of asset prices (see Cont and Tankov [9] for a discussion of Lévy processes in finance). In particular, some processes like the normal inverse Gaussian (NIG) or the hyperbolic Lévy process have become particularly relevant since they are able to capture the return distribution of most asset prices (see e.g. Barndorff-Nielsen [3] and Eberlein and Keller [11]). These Lévy processes are pure-jump, and therefore give distinctively different paths of the asset prices compared to a Brownian motion with continuous paths. In empirical analysis of financial price data, one may detect big jumps, however, the small jumps are very hard to separate from the observations of a Brownian motion. Thus, it is not a simple task to decide whether a Lévy process with jumps or a Brownian motion is governing the small variations in a stock price, say.

In this paper we focus on the stability of Merton's problem with respect to model choice. In particular, we analyse what happens when the small jumps of the Lévy process driving the asset price dynamics is approximated by a Brownian motion. This would mimic a situation where we have two investors, one believing in a pure-jump Lévy process, and another which thinks the small variations in prices come from a Brownian motion. Asmussen and Rosinski [2] show that in fact the small jumps of a Lévy process has a central limit type behaviour towards a Brownian motion, which tells that one may empirically not be able to distinguish between two such models. The question is then to what extent this transfer over the the optimal portfolio selection problem. We pose the problem as an approximation of asset price models, where we either ignore or subsitute jumps in the Lévy process smaller than a threshold ϵ. To substitute, we use a Brownian motion. Indeed, our analysis shows that the optimal investment in the risky asset is stable with respect to the different approximations. We are able to classify the convergence rate as being proportional to the variance of the small jump part of the Lévy process.

A general approach to stability of stochastic control problems are provided by Larsen and Žitković [16]. They investigate the influence of estimation errors in the parameters of the underlying financial assets. Jakobsen, Karlsen and La Chioma [14] are deriving stability results for the Hamilton-Jacobi-Bellman equation for stochastic control problems, and derive error estimates for approximative viscosity solutions. In a paper by Benth, Di Nunno and Khedher [4], stability for option pricing and hedging have been considered based on similar Lévy approximations as in the present paper. Here, the authors prove that prices and hedges converge at a rate given by the variance of the small jumps of the Lévy process, similar to our findings.

Our results are presented as follows. In Section 2 we state the control problems and recall some results on these. Afterwards we discuss in Section 3 how the approximation of the Lévy process influences the integral equation which gives implicitly the solution of the control problem. In Section 4 we study the convergence of the controls and derive convergence rates, which is illustrated by some numerical examples. The convergence of the value functions is treated in Section 5, and in Section 6 we analyse the wealth processes.

2. A review of Merton's portfolio optimization problem

We recall the Merton's portfolio optimization problem in the Lévy case with and without consumption, and review some relevant results from Benth et al. [5, 6].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ a given filtration satisfying the usual conditions. We consider a financial market consisting of a stock and a bond. Let the bond dynamics be given by

$$dB(t) = rB(t)dt\,,$$

where $r > 0$ is the constant interest rate. The value of the stock follows a process given by

$$S(t) = S(0)e^{\xi t + L(t)}$$

where ξ is a constant and $L(t)$ a pure jump Lévy process with Lévy-Khintchine decomposition

$$L(t) = \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(dt, dz).$$

Here, $N(dt, dz)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$ with intensity measure $dt \times \nu(dz)$, and $\nu(dz)$ being the Lévy measure, that is, a σ-finite Borel measure on $\mathbb{R} \setminus \{0\}$ with

$$\int_{\mathbb{R}\setminus\{0\}} \min(1, z^2)\, \nu(dz) < \infty\,.$$

We denote by $\tilde{N}(ds, dz) = N(dt, dz) - dt\nu(dz)$ the compensated Poisson random measure of N. In the sequel, we suppose that the Lévy process is exponentially integrable, that is, its Lévy measure satisfies the condition

$$\int_1^\infty e^{2z}\, \nu(dz) < \infty\,. \tag{1}$$

This will ensure that the stock price dynamics has finite expectation, but also it will be necessary for the analysis to come in order to derive convergence rates for approximative portfolio strategies.

Consider an investor who puts her money in the stock and bond to optimize her utility. Let $\pi(t)$ denote the fraction of her wealth invested in the stock and $c(t)$ her rate of consumption at time t. The dynamics of the wealth $X^{(\pi,c)}(t)$ becomes (see Benth et al. [5])

$$dX^{(\pi,c)}(t) = X^{(\pi,c)}(t)(r + (\hat{\mu} - r)\pi(t))dt - c(t)\, dt + X^{(\pi,c)}(t-)\pi(t-) \int_{\mathbb{R}\setminus\{0\}} e^z - 1\tilde{N}(dt, dz), \tag{2}$$

where

$$\hat{\mu} = \xi + \int_{\mathbb{R}\setminus\{0\}} e^z - 1 - z1_{|z|<1}\nu(dz)$$

is the drift of the stock price dynamics and $X(t-)$ denotes the left-limit of a process $X(t)$. We denote by $X^{(\pi,c)}(0) = x$ the initial wealth of the investor, and assume $r < \hat{\mu}$. The last conditions ensures that the stock gives a higher average return than the bond.

We define the set of *admissible controls* $\mathcal{A}_x$ to consist of those investment-consumption plans (π, c) such that

(1) $\pi(t)$ is progressively measurable with values in $[0, 1]$,
(2) $c(t)$ is a positive and adapted process such that $\int_0^t \mathbb{E}[c(s)]ds < \infty$ for all $t \geq 0$,
(3) $c(t)$ is such that $X^{(\pi,c)}(t) \geq 0$ almost everywhere for all $t \geq 0$.

We will restrict our attention to admissible controls, $(\pi, c) \in \mathcal{A}_x$. Observe that we constrain the invested fraction of wealth in the stock to be between 0 and 1, meaning that we cannot short sell stocks or borrow money to invest more that our wealth in stocks. One may extend the theory in Benth et al. [5] to $\pi \in [\underline{\pi}, \overline{\pi}]$, for $\underline{\pi} < 0$ and $\overline{\pi} > 0$. In the considerations to come, we can also include such a case with some additional effort.

The utility derived by the investor comes from consumption, and we suppose that she has a power utility function of HARA type, that is, $U(x) = x^\gamma/\gamma$ for a risk aversion parameter $\gamma \in (0, 1)$. Letting $\delta > 0$ be a constant discount rate, the value function is defined by

$$V(x) = \sup_{c,\pi\in\mathcal{A}_x} \mathbb{E}^x\Big[\int_0^\infty e^{-\delta t}\big[\frac{c_t^\gamma}{\gamma}\big]dt\Big]\,. \tag{3}$$

By dynamic programming, the Hamilton-Jacobi-Bellman (HJB) equation takes the form

$$\begin{aligned}\max_{c\geq 0,\pi\in[0,1]} &\big[(r + (\hat{\mu} - r)\pi_t)xv'(x) - cv'(x) - \delta v(x) + \frac{c^\gamma}{\gamma} \\ &+ \int_{\mathbb{R}\setminus\{0\}} \big(v(x + \pi x(e^z - 1)) - v(x) - \pi x v'(x)(e^z - 1)\big)\nu(dz)\big] = 0\,.\end{aligned} \tag{4}$$

Benth et al. [5] show that V is a viscosity solution of the HJB-equation. Moreover, the optimal investment strategy turns out to be a constant π^* solving implicitly the integral equation

$$\int_{\mathbb{R}\setminus\{0\}} (1 + \pi^*(e^z - 1))^{\gamma-1}(e^z - 1) - (e^z - 1)\nu(dz) = r - \hat{\mu}\,. \tag{5}$$

The optimal consumption is given as a constant rate of the wealth,

$$c^*(t) = X^{(\pi,c)}(t)\frac{1-\gamma}{\delta - k(\gamma)} \tag{6}$$

where

$$k(\gamma) = \gamma(r + (\hat{\mu} - r)\pi^*) + \int_{\mathbb{R}\setminus\{0\}} (1 + \pi^*(e^z - 1))^\gamma - 1 - \gamma\pi(e^z - 1)\nu(dz).$$

We remark in passing that one may consider the simplified problem of maximizing terminal wealth only, and not consume anything from the portfolio. The value function becomes in this case

$$V(x) = \sup_{\pi \in \mathcal{A}_x} \mathbb{E}^x\left[\frac{1}{\gamma}X^\pi(T)^\gamma\right], \tag{7}$$

where we use the obvious definition of the set of admissible controls and the wealth process $X^\pi(t)$ (the latter is given by $X^{(\pi,0)}(t)$). As it turns out, the optimal investment strategy is still a constant fraction of wealth placed in the stock, solving the integral equation (5).

We end this section with a discussion on conditions ensuring the existence and uniqueness of an optimal portfolio investment strategy $\pi^* \in [0, 1]$. For this purpose, define

$$F(\pi) = \int_{\mathbb{R}\setminus\{0\}} (1 + \pi(e^z - 1))^{\gamma-1}(e^z - 1) - (e^z - 1)\nu(dz) + (\hat{\mu} - r)\,, \tag{8}$$

which is a continuous function on $[0, 1]$ under our exponential integrability hypothesis on $\nu(dz)$. It holds

$$F(0) = \hat{\mu} - r\,,$$

which is positive by assumption on $\hat{\mu}$ and r. We have:

Lemma 1. *Assume that the Lévy measure and γ satisfy*

$$\int_{\mathbb{R}\setminus\{0\}} (e^z - 1)(1 - e^{-(1-\gamma)z})\,\nu(dz) > \hat{\mu} - r\,. \tag{9}$$

Then there exists a unique $\pi^* \in (0, 1)$ solving (5).

Proof. By commuting differentiation and integration (see Folland [13]), we find for F in (8) that

$$F'(\pi) = -(1-\gamma)\int_{\mathbb{R}\setminus\{0\}} (e^z - 1)^2(1 + \pi(e^z - 1))^{\gamma-2}\,\nu(dz)\,.$$

Since $(\exp(z) - 1)^2$ and $(1 + \pi(\exp(z) - 1))^{\gamma-2}$ are both positive as long as $\pi \in [0, 1]$, we find that $F'(\pi) < 0$. Hence, F is strictly decreasing on $[0, 1]$. Therefore, we have a unique solution $\pi^* \in (0, 1)$ of (5) as long as $F(1) < 0$. But this is ensured by the condition in the Lemma. □

Note that $(\exp(z) - 1)(1 - \exp(-(1-\gamma)z))$ is positive for all $z \in \mathbb{R}$. Hence, the left-hand side of the condition (9) is positive. Hence, the condition therefore gives a relation between the Lévy measure and the risk aversion on one hand, and the excess return $\hat{\mu} - r$ on the other. Given the optimal control π^*, we have the optimal consumption process $c^*(t)$ as well.

For the analysis to come, it is convenient to introduce a function $f(\pi, z)$ defined as

$$f(\pi, z) = (1 + \pi(e^z - 1))^{\gamma-1}(e^z - 1) - (e^z - 1)\,. \tag{10}$$

Furthermore, let $g(\pi)$ be

$$g(\pi) = \int_{\mathbb{R}\setminus\{0\}} f(\pi, z)\nu(dz)\,. \tag{11}$$

Then, from the definition of $F(\pi)$ we see that the integral equation (5) may be formulated compactly as

$$g(\pi) = r - \hat{\mu}. \tag{12}$$

We shall make use of these two functions when we move on in the next Section to consider approximations of the control problem of Merton.

3. The control problem with approximated driving process

In this section we examine the convergence properties of Merton's portfolio problem when we approximate the Lévy process $L(t)$ in the stock price dynamics. In particular, we consider two approximations, one where the small jumps of L are neglected, and another where we substitute the small jumps by a scaled Brownian motion. These two approximations will lead to different HJB-equations, and thus to different controls and value functions. We analyse the convergence to the original portfolio problem, and establish rates.

3.1. **Approximating L by neglecting the small jumps.** By appealing to the Lévy-Kintchine representation of $L(t)$, we can write for a given $0 < \epsilon < 1$,

$$L(t) = \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(ds, dz) \tag{13}$$

$$= \int_0^t \int_{|z|<\epsilon} z\tilde{N}(ds, dz) + \int_0^t \int_{\epsilon<|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(dt, dz). \tag{14}$$

Introduce an approximation of $L(t)$ which neglects jumps smaller than ϵ:

$$L_{N,\epsilon}(t) = \int_0^t \int_{\epsilon<|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(dt, dz).$$

Then, the Levy measure $\nu_{N,\epsilon}$ of $L_{N,\epsilon}(t)$ is

$$\nu_{N,\epsilon}(dz) := \begin{cases} \nu(dz)\,, & |z| > \epsilon \\ 0\,, & \text{otherwise.} \end{cases}$$

The neglection of the small jumps influences the Levy measure and so indirectly also all terms which include a Lévy integral, as the drift of the stock price dynamics, the optimal controls and eventually the value function.

With obvious definition, we denote by $X_{N,\epsilon}^{(\pi,c)}(t)$ the wealth process for admissible controls $(\pi, c) \in \mathcal{A}_{x,N,\epsilon}$. Furthermore, $V_{N,\epsilon}(x)$ denotes the value function.

Tracing through the derivation of Benth et al. [5] using $L_{N,\epsilon}$ in the stock price dynamics, leads to the following integral equation for the optimal control $\pi^*_{N,\epsilon}$

$$\int_{\mathbb{R}\setminus\{0\}} f(\pi^*_{N,\epsilon}, z)\nu_{N,\epsilon}(dz) = r - \hat{\mu}_{N,\epsilon} \tag{15}$$

with

$$\begin{aligned} \hat{\mu}_{N,\epsilon} &= \xi + \int_{|z|>\epsilon} e^z - 1 - z1_{|z|<1}\,\nu(dz) \\ &= \hat{\mu} - \int_{|z|<\epsilon} e^z - 1 - z\,\nu(dz)\,. \end{aligned}$$

Furthermore, the optimal consumption process, denoted $c^*_{N,\epsilon}(t)$ will be the same as for the non-approximated case, except that we insert the optimal control $\pi^*_{N,\epsilon}$ in (6) and use $\nu_{N,\epsilon}$ as the Lévy measure in the definition of k.

Let us investigate conditions for the existence of a unique solution to (15). Introduce the function $F_{N,\epsilon}(\pi)$ as

$$F_{N,\epsilon}(\pi) = \int_{\mathbb{R}\setminus\{0\}} (1 + \pi(e^z - 1))^{\gamma-1}(e^z - 1) - (e^z - 1)\,\nu_{N,\epsilon}(dz) + (\hat{\mu}_{N,\epsilon} - r)\,. \tag{16}$$

The optimal investment strategy $\pi^*_{N,\epsilon}$ is given as a root of the function $F_{N,\epsilon}$. Observe that the derivative of $F_{N,\epsilon}$ is negative, similar as to the case of $\epsilon = 0$. Thus, $F_{N,\epsilon}$ is a continuous function which is strictly decreasing. It will have a root in the interval $[0, 1]$ if and only if $F_{N,\epsilon}(0) > 0$ and $F_{N,\epsilon}(1) < 0$. But this is equivalent to

$$\hat{\mu} - r > \int_{|z|<\epsilon} e^z - 1 - z\,\nu(dz) \tag{17}$$

and

$$\int_{\mathbb{R}\setminus\{0\}} (e^z - 1)(1 - e^{-(1-\gamma)z})\,\nu_{N,\epsilon}(dz) < \hat{\mu} - r - \int_{|z|<\epsilon} e^z - 1 - z\,\nu(dz)\,. \tag{18}$$

We note that by Taylor expansion, the integral

$$\int_{|z|<\epsilon} e^z - 1 - z\,\nu(dz)\,,$$

will be approximately equal to $\sigma^2(\epsilon)$ which tends to zero as $\epsilon \to 0$. Recalling that $\hat{\mu} > r$, we are ensured the existence and uniqueness of a solution $\pi^*_{N,\epsilon}$ by choosing ϵ sufficiently small if the condition (9) in Lemma 1 holds (that is, the condition for existence and uniqueness of $\pi^* \in [0,1]$).

To emphasize the difference of (15) from the original equation (5), reorganize to show that (15) is equivalent to

$$\int_{\mathbb{R}\setminus\{0\}} f(\pi^*_{N,\epsilon}, z)\nu(dz) - \int_{|z|<\epsilon} f(\pi^*_{N,\epsilon}, z)\,\nu(dz) = r - (\hat{\mu} - \int_{|z|<\epsilon} e^z - 1 - z\,\nu(dz))\,.$$

Or, using the function g in (11), we have

$$g(\pi^*_{N,\epsilon}) = r - \hat{\mu} + \int_{|z|<\epsilon} \left(1 + \pi^*_{N,\epsilon}(e^z - 1)\right)^{\gamma-1}(e^z - 1) - z\,\nu(dz).$$

Introduce the function $h(\pi, z)$ for $|z| < 1$ by

$$h(\pi, z) := (1 + \pi(e^z - 1))^{\gamma-1}(e^z - 1) - z\,. \tag{19}$$

Then, it finally follows that $\pi^*_{N,\epsilon}$ is the solution of the integral equation

$$g(\pi^*_{N,\epsilon}) = r - \hat{\mu} + \int_{|z|<\epsilon} h(\pi^*_{N,\epsilon}, z)\,\nu(dz). \tag{20}$$

In the analysis of convergence of $\pi^*_{N,\epsilon}$ to π^* as $\epsilon \to 0$, this representation of the optimal control is attractive.

3.2. **Approximating L by substituting small jumps by Brownian motion.** An alternative to truncating off the small jumps, is to approximate them by an appropriately scaled Brownian motion as discussed Asmussen and Rosinski [2]. More precisely, we introduce the process

$$L_{W,\epsilon}(t) = \sigma(\epsilon)W(t) + L_{N,\epsilon}(t)\,, \tag{21}$$

where $W(t)$ is a Brownian motion (independent of $L(t)$) and

$$\sigma^2(\epsilon) := \int_{|z|<\epsilon} z^2\nu(dz)\,, \tag{22}$$

is the variance of the small jumps (at least for symmetric Lévy processes). Note that $\sigma^2(\epsilon)$ is finite since $\nu(dz)$ integrates z^2 around the origin by definition. Moreover, by monotone convergence, it holds that

$$\lim_{\epsilon\to 0} \sigma^2(\epsilon) = 0\,.$$

It will be clear later that $\sigma^2(\epsilon)$ gives the rate of convergence in the approximations of the original portfolio optimization problem.

As in Benth et al. [6], an additional Brownian component does not change the general form of the solution of the control problem. We denote the wealth equation by $X^{(\pi,c)}_{W,\epsilon}$ for admissible controls $(\pi, c) \in \mathcal{A}_{x,W,\epsilon}$, with an obvious definition of these. The value function in this case is denoted $V_{W,\epsilon}(x)$.

We can derive an integral equation for the optimal investment strategy, still being a constant $\pi^*_{W,\epsilon}$, but now solving the integral equation

$$\int_{\mathbb{R}\setminus\{0\}} f(\pi^*_{W,\epsilon}, z)\nu_{N,\epsilon}(dz) = r - \hat{\mu}_{W,\epsilon} + (1-\gamma)\sigma^2(\epsilon)\pi^*_{W,\epsilon}\,,$$

with

$$\begin{aligned}\hat{\mu}_{W,\epsilon} &= \xi + \int_{|z|>\epsilon} e^z - 1 - z 1_{|z|<1}\,\nu(dz) + \frac{1}{2}\sigma^2(\epsilon)\\ &= \hat{\mu} - \int_{|z|<\epsilon} e^z - 1 - z\,\nu(dz) + \frac{1}{2}\sigma^2(\epsilon).\end{aligned}$$

The optimal consumption $c^*_{W,\epsilon}(t)$ is given by

$$c^*_{W,\epsilon}(t) = X^{(\pi^*_{W,\epsilon},c)}(t)\frac{1-\gamma}{\delta - k_{W,\epsilon}(\gamma)} \tag{23}$$

where

$$\begin{aligned}k_{W,\epsilon}(\gamma) &= \gamma(r + (\hat{\mu}_{W,\epsilon} - r)\pi^*_{W,\epsilon}) - \frac{1}{2}\sigma^2(\epsilon)(\pi^*_{W,\epsilon})^2\gamma(\gamma-1)\\ &\quad + \int_{\mathbb{R}\setminus\{0\}} (1 + \pi^*_{W,\epsilon}(e^z-1))^\gamma - 1 - \gamma\pi^*_{W,\epsilon}(e^z-1)\nu_{N,\epsilon}(dz)\,.\end{aligned}$$

Again, we reformulate the equation for the optimal investment strategy in terms of g, in order to find

$$g(\pi^*_{W,\epsilon}) = r - \hat{\mu} + \sigma^2(\epsilon)\left((1-\gamma)\pi^*_{W,\epsilon} - \frac{1}{2}\right) + \int_{|z|<\epsilon} h(\pi^*_{W,\epsilon}, z)\,\nu(dz)\,. \tag{24}$$

Additionally to $\int_{|z|<\epsilon} h(\pi^*_{W,\epsilon}, z)\,\nu(dz)$, that also appeared in (20), we have a term with $\sigma^2(\epsilon)$ on the right hand side of (24).

We state conditions for the existence and uniqueness of an optimal strategy $\pi^*_{W,\epsilon} \in [0,1]$. Define the function $F_{W,\epsilon}(\pi)$ as

$$F_{W,\epsilon}(\pi) = \int_{\mathbb{R}\setminus\{0\}} (1+\pi(e^z-1))^{\gamma-1}(e^z-1) - (e^z-1)\,\nu_{N,\epsilon}(dz) + (\hat{\mu}_{W,\epsilon} - r) - (1-\gamma)\sigma^2(\epsilon)\pi\,. \tag{25}$$

Similar to the case of neglecting the small jumps, $F_{W,\epsilon}(\pi)$ is a strictly decreasing continuous function, which has a root in the interval $[0,1]$ if and only if $F_{W,\epsilon}(0) > 0$ and $F_{W,\epsilon}(1) < 0$. This is equivalent to

$$\hat{\mu} - r > \int_{|z|<\epsilon} e^z - 1 - z\,\nu(dz) - \frac{1}{2}\sigma^2(\epsilon) \tag{26}$$

and

$$\int_{\mathbb{R}\setminus\{0\}} (e^z-1)(1-e^{-(1-\gamma)z})\,\nu_{N,\epsilon}(dz) < \hat{\mu} - r - \int_{|z|<\epsilon} e^z - 1 - z\,\nu(dz) - (\frac{1}{2} - \gamma)\sigma^2(\epsilon)\,. \tag{27}$$

For the same reasons as before, we are ensured the existence and uniqueness of a solution $\pi^*_{W,\epsilon}$ by choosing ϵ sufficiently small if the condition (9) in Lemma 1 holds.

Let us discuss an example. In finance, the normal inverse Gaussian (NIG) distribution turns out to model the (log-)returns of financial asset prices very well. There exist several empirical studies where this distribution was applied, but we refer to Bølviken and Benth [7] which studied Norwegian stock prices. The NIG Lévy process is a pure-jump process, where the Lévy measure is explicitly known as

$$\nu(dz) = \frac{\alpha\delta}{\pi|z|}K_1(\alpha|z|)e^{\beta z}\,dz$$

for a NIG$(\mu, \beta, \alpha, \delta)$ process. Here, K_1 is the modified Bessel function of the third kind of index 1. In Rydberg [18] it was suggested to approximate the small jumps of the NIG process by a Brownian motion scaled by $\sigma(\epsilon)$, as discussed above. In Asmussen and Rosinski [2] this example was further elaborated, and they show that

$$\sigma^2(\epsilon) \sim \frac{2\delta}{\pi} \times \epsilon\,.$$

We remark that the NIG Lévy process was also used in Benth et al. [5] as a motivation for their studies of the Merton portfolio optimization problem for pure-jump Lévy processes.

4. Convergence rates for the optimal investment strategy

In this Section we prove that the approximative investment strategies $\pi^*_{N,\epsilon}$ and $\pi^*_{W,\epsilon}$ both converge to π^* as $\epsilon \to 0$. Moreover, we derive rates of convergence for both approximations in terms of the variance of the small jumps $\sigma^2(\epsilon)$.

4.1. **Approximation of L by neglecting small jumps.** Consider the case where we derive the optimal portfolio strategy $\pi^*_{N,\epsilon}$ based on an approxmation where the small jumps are simply neglected. We have the following result.

Proposition 1. *The control $\pi^*_{N,\epsilon}$ solving* (15) *converges to the control π^* derived from* (5) *when $\epsilon \to 0$. In particular, it holds*

$$|\pi^*_{N,\epsilon} - \pi^*| \leq C_N \sigma^2(\epsilon)\,,$$

for a constant $C_N > 0$ independent of ϵ.

Proof. Recall the definition of the function f in (10) to see that

$$\frac{\partial}{\partial \pi} f(\pi, z) := f'(\pi, z) = (\gamma - 1)(e^z - 1)^2\left(1 + \pi(e^z - 1)\right)^{\gamma - 2}$$

is negative for all $z \in \mathbb{R}$. Hence, f is a continuous and strictly decreasing function of $\pi \in [0, 1]$. Moreover, it is negative for all $\pi \in (0, 1),\ z \in \mathbb{R}$ since $f(0) = 0$. Then, it follows that $g(\pi) = \int_{\mathbb{R}\setminus\{0\}} f(\pi, z)\nu(dz)$ in (11) is also strictly decreasing and negative. As

$$|f(\pi, z)| \leq |f(1, z)| \text{ and } \int_{\mathbb{R}\setminus\{0\}} |f(1, z)|\nu(dz) < \infty\,,$$

the parameter-dependent integral defining $g(\pi)$ is continuous by Theorem 11.4 in Schilling [19]. Therefore, the inverse g^{-1} exists and is continuous on the image $g([0, 1])$, and we can write the optimal control as

$$\pi^* = g^{-1}(r - \hat{\mu}).$$

Moreover, from elementary calculus the derivative of the inverse of a function $g(\pi) = y$ can be written as

$$\begin{aligned}(g^{-1})'(y) &= \frac{1}{g'(g^{-1}(y))} = \frac{1}{g'(\pi)} \\ &= \frac{1}{\frac{\partial}{\partial \pi}\int f(\pi, z)\,\nu(dz)} = \frac{1}{\int \frac{\partial}{\partial \pi} f(\pi, z)\,\nu(dz)}\end{aligned}$$

where we are allowed to commute integration and differentiation using Theorem 11.5 in Schilling [19] as long as $|\frac{\partial}{\partial \pi} f(\pi, z)| \leq w(z)$ for $w(z)$ being an integrable function. But for $z > 0$ we find that

$$|\frac{\partial f}{\partial \pi}| \leq (1 - \gamma)(e^z - 1)^2$$

whereas for $z < 0$ we find

$$|\frac{\partial f}{\partial \pi}| \leq (1 - \gamma)(e^z - 1)^2 e^{-(2-\gamma)z}\,.$$

By the exponential integrability hypothesis on $\nu(dz)$, this defines an integrable function $w(z)$ verifying the commuting of integration of differentiation.

We continue with the proof of convergence. For $z > 0$, $\partial f/\partial \pi$ is monotonely increasing in $\pi \in [0, 1]$ and for $z < 0$ it is monotonely decreasing. Additionally, $\partial f/\partial \pi$ is negative for all $z \in \mathbb{R}$. So,

$$\begin{aligned}\frac{\partial}{\partial \pi} f(\pi, z) &\leq \frac{\partial}{\partial \pi} f(0, z),\ z < 0 \\ \frac{\partial}{\partial \pi} f(\pi, z) &\leq \frac{\partial}{\partial \pi} f(1, z),\ z > 0.\end{aligned}$$

Next we apply this to find an approximation for $(g^{-1})'(y)$:

$$\begin{aligned}\int_{\mathbb{R}\setminus\{0\}} \frac{\partial f}{\partial \pi}(\pi,z)\nu(dz) &= \int_{\mathbb{R}^+} \frac{\partial f}{\partial \pi}(\pi,z)\nu(dz) + \int_{\mathbb{R}^-} \frac{\partial f}{\partial \pi}(\pi,z)\nu(dz)\\ &\le \int_{\mathbb{R}^+} \frac{\partial f}{\partial \pi}(1,z)\nu(dz) + \int_{\mathbb{R}^-} \frac{\partial f}{\partial \pi}(0,z)\nu(dz)\end{aligned}$$

where we have

$$\begin{aligned}&\frac{\partial f}{\partial \pi}(0,z) = (\gamma-1)(e^z-1)^2\\ &\frac{\partial f}{\partial \pi}(1,z) = (\gamma-1)(e^z-1)^2 e^{z(\gamma-2)}\,.\end{aligned}$$

Hence,

$$\begin{aligned}\Big|\int_{\mathbb{R}\setminus\{0\}} f'(\pi,z)\nu(dz)\Big| &\ge \Big|\int_{\mathbb{R}^+} f'(1,z)\nu(dz) + \int_{\mathbb{R}^-} f'(0,z)\nu(dz)\Big|\\ &= (1-\gamma)\left\{\int_0^\infty (e^z-1)^2 e^{-(2-\gamma)z}\,\nu(dz) + \int_{-\infty}^0 (e^z-1)^2\,\nu(dz)\right\}\\ &=: L^{-1}\,.\end{aligned}$$

This implies that

$$|(g^{-1})'(y)| = \frac{1}{\big|\int_{\mathbb{R}\setminus\{0\}} f'(\pi,z)\nu(dz)\big|} \le L\,.$$

With this bound on the derivative of the inverse function, we move on to estimate the error:

By applying the Mean Value Theorem in calculus to the equations (12) and (20), we find

$$\begin{aligned}|\pi^*_{N,\epsilon}-\pi^*| &= \left|g^{-1}\left(r-\hat\mu+\int_{|z|<\epsilon} h(\pi^*_{N,\epsilon},z)\nu(dz)\right) - g^{-1}(r-\hat\mu)\right|\\ &= |(g^{-1})'(\theta)|\left|r-\hat\mu+\int_{|z|<\epsilon} h(\pi^*_{N,\epsilon},z)\nu(dz)-(r-\hat\mu)\right|\\ &\le L\Big|\int_{|z|<\epsilon} h(\pi^*_{N,\epsilon},z)\nu(dz)\Big|\end{aligned}$$

for some

$$\theta \in [\min\{r-\hat\mu, r-\hat\mu+\int_{|z|<\epsilon} h(\pi^*_{N,\epsilon},z)\,\nu(dz)\}, \max\{r-\hat\mu, r-\hat\mu+\int_{|z|<\epsilon} h(\pi^*_{N,\epsilon},z)\,\nu(dz)\}]\,.$$

Next, let us estimate the term involving h. Since $h(\pi,z)$ is decreasing in $\pi\in[0,1]$, we have

$$h(0,z)\ge h(\pi,z)\ge h(1,z)\,.$$

Note that h can be both positive and negative, which makes it difficult for estimations of the absolute value of the integral of h. Divide the domain of definition of h in z into those parts where h is positive and negative for fixed π and γ:

$$\begin{aligned}A_\pi &:= \{z\in\mathbb{R} : h(\pi,z)\ge 0\}\,,\\ B_\pi &:= \{z\in\mathbb{R} : h(\pi,z)< 0\}\,.\end{aligned}$$

Then, for $z\in A_\pi$, we find from Taylor expansions

$$|h(\pi,z)|\le|h(0,z)| = |e^z-1-z| \le \sum_{n=2}^\infty \frac{|z|^n}{n!} \le z^2 e^{|z|}\,,$$

and for $z \in B_\pi$ we have

$$
\begin{aligned}
|h(\pi,z)| \le |h(1,z)| &= |e^{z(\gamma-1)}(e^z-1)-z| = |e^{\gamma z} - e^{z(\gamma-1)} - z| \\
&\le \sum_{n=2}^{\infty} \frac{|\gamma^n - (\gamma-1)^n|}{n!}|z|^n \le z^2 \sum_{n=0}^{\infty} \frac{|\gamma^{n+2} - (\gamma-1)^{n+2}|}{n!}|z|^n \le z^2 e^{|z|}\,,
\end{aligned}
$$

as $|\gamma^{n+2} - (\gamma-1)^{n+2}| \le 1$. It follows

$$
\begin{aligned}
|\pi^*_{N,\epsilon} - \pi^*| &\le L\Big|\int_{|z|<\epsilon} h(\pi^*_{N,\epsilon},z)\,\nu(dz)\Big| \\
&\le L\Big(\Big|\int_{\{|z|<\epsilon\}\cap A_{\pi^*_{N,\epsilon}}} h(\pi^*_{N,\epsilon},z)\,\nu(dz)\Big| + \Big|\int_{\{|z|<\epsilon\}\cap B_{\pi^*_{N,\epsilon}}} h(\pi^*_{N,\epsilon},z)\,\nu(dz)\Big|\Big) \\
&\le L\Big(\int_{\{|z|<\epsilon\}\cap A_{\pi^*_{N,\epsilon}}} |h(\pi^*_{N,\epsilon},z)|\,\nu(dz) + \int_{\{|z|<\epsilon\}\cap B_{\pi^*_{N,\epsilon}}} |h(\pi^*_{N,\epsilon},z)|\,\nu(dz)\Big) \\
&\le Le^{\epsilon}\Big(\int_{\{|z|<\epsilon\}\cap A_{\pi^*_{N,\epsilon}}} z^2\,\nu(dz) + \int_{\{|z|<\epsilon\}\cap B_{\pi^*_{N,\epsilon}}} z^2\nu(dz)\Big) \\
&\le Le^{\epsilon}\sigma^2(\epsilon)\,.
\end{aligned}
$$

This completes the proof. □

Remark that since $h(1,z)$ is increasing in $z \in [-1,1]$, we find that

$$h(1,z) \ge h(1,-1) = e^{-\gamma}(1-e) + 1\,.$$

It follows that $h(\pi,z)$ is positive for all $z \in \mathbb{R}$ and $\pi \in [0,1]$ if

$$\gamma > \ln(e-1) \approx 0.541. \tag{28}$$

Thus, if $\gamma \ge \ln(e-1)$, we have

$$|h(\pi,z)| \le |h(0,z)| = e^z - 1 - z\,.$$

This implies that

$$|\int_{\mathbb{R}_0} h(\pi,z)\,\nu(dz)| \le e^{\epsilon}\sigma^2(\epsilon)$$

which simplies the proof above.

Going back to the case of an NIG Lévy process $L(t)$, then by neglecting the small jumps and solving the portfolio optimization problem would yield an error which could be bounded as

$$|\pi^*_{N,\epsilon} - \pi^*| \le C \times \epsilon\,.$$

Thus, the speed of convergence is of order 1 with respect to the truncation error ϵ.

4.2. **Approximation of L by substituting small jumps by Brownian motion.** We move on showing that the approximation using a Brownian motion leads to a convergence of $\pi^*_{W,\epsilon}$ to π^* with the same rate as for the case where small jumps are neglected. We formulate the result as a proposition:

Proposition 2. *The control $\pi^*_{W,\epsilon}$ solving* (24) *converges to the control π^* derived from* (12) *when $\epsilon \to 0$. In particular, it holds*

$$|\pi^*_{W,\epsilon} - \pi^*| \le C_W\sigma^2(\epsilon)\,,$$

for a constant $C_W > 0$ independent of ϵ.

Proof. The proof follows the same line of arguments as in the proof of Prop. 1.

$$\begin{aligned}|\pi^*_{W,\epsilon} - \pi^*| &= \big|g^{-1}(r - \hat{\mu} + \sigma^2(\epsilon)\big((1-\gamma)\pi^*_{W,\epsilon} - \frac{1}{2}\big) + \int_{|z|<\epsilon} h(\pi^*_{W,\epsilon}, z)\,\nu(dz)) - g^{-1}(r-\hat{\mu})\big| \\ &\leq L\big|r - \hat{\mu} + \sigma^2(\epsilon)\big((1-\gamma)\pi^*_{W,\epsilon} - \frac{1}{2}\big) - (r-\hat{\mu})\big| + \big|\int_{|z|<\epsilon} h(\pi^*_{W,\epsilon}, z)\,\nu(dz)\big| \\ &\leq L\Big(\sigma^2(\epsilon)\big(\frac{3}{2} - \gamma\big) + \big|\int_{|z|<\epsilon} h(\pi_{W,\epsilon}, z)\nu(dz)\big|\Big)\,.\end{aligned}$$

Invoking the estimations on h from the proof of Prop. 1 gives the result. □

Inspecting the proofs of Prop. 1 and 2 shows that the constant in the convergence rate when neglecting the small jumps is given by $C_N = Le$, whereas for the Brownian motion approximation it is $C_W = L(3/2-\gamma) + C_N > C_N$. Thus, the error estimate is in fact slightly worse when we use an approximation which gives a Lévy process with approximately the same variance, compared to an approximation where some of the noise is removed.

4.3. **Examples.** First, let us assume the driving process $L(t)$ is a Poisson process $N(t)$, compensated by its jump intensity λ,

$$L(t) = N(t) - \lambda t\,.$$

This is admittedly not a process which has "small jumps", since all jumps are of constant size 1. However, we would like to look at an example where we perturb this process by adding a Brownian motion component, that is

$$L_\epsilon(t) = L(t) + \epsilon W(t)\,.$$

This is not fitting into our analysis above, but the case here is to see the effect of perturbing the process $L(t)$ in a simple setting where much is known analytically. It serves as a "non-example" which is still relevant for our considerations.

We find the optimal control π^* by solving (12), which in this case becomes

$$((1+\pi^*(e-1))^{\gamma-1}(e-1) - (e-1))\lambda = r - ((\gamma-\lambda) + (e-1)\lambda)\,.$$

This yields the solution

$$\pi^* = \left(\left(\frac{r-\gamma+\lambda}{(e-1)\lambda}\right)^{\frac{1}{\gamma-1}} - 1\right)\frac{1}{e-1}.$$

The corresponding equation for solving π^*_ϵ, the optimal fraction to invest in the stock for the driving process $L_\epsilon(t)$, is

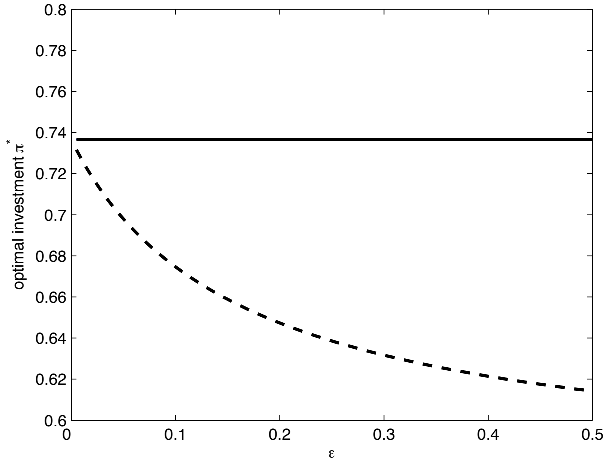
$$((1+\pi^*_\epsilon(e-1))^{\gamma-1}(e-1) - (e-1))\lambda = r - ((\gamma-\lambda) + (e-1)\lambda) + ((1-\gamma)\pi^*_\epsilon - \frac{1}{2})\epsilon\,,$$

which we solve numerically.

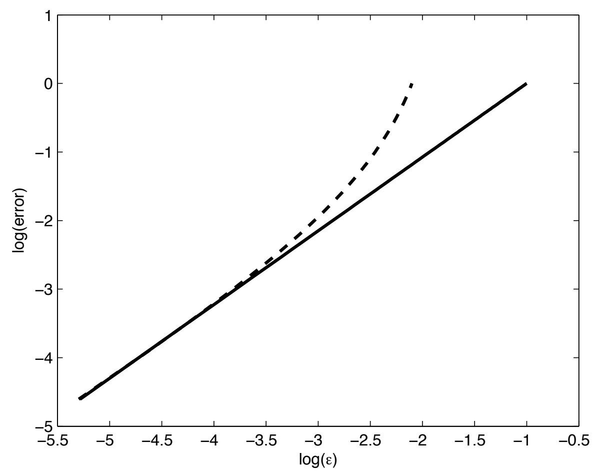
To be concrete, we suppose an annual interest rate $r = 4.5\%$ and a jump intensity given by $\lambda = 0.5$, corresponding to an asset that jumps on average every second day. Furthermore, the risk aversion is set equal to $\gamma = 0.5$. This yields an optimal investment in the stock of $\pi^* = 0.7367$, that is, 73.67% of the wealth should go into the stock. In Figure 1(a) we plot the "exact solution" π^* against the approximative π^*_ϵ for ϵ ranging from 0.01 to 1. The approximative investment strategy is found by solving the equation using the built-in Matlab routine `fzero`. Since the approximative model has more noise/uncertainty, it naturally leads to an investment strategy less than π^*. Noteworthy is that the difference is rather big even for small ϵ's. For example, if $\epsilon = 0.1$, we have a relative error of approximately 5.3%, whereas for $\epsilon = 0.01$ it is 0.7%.

We see from the figure where we plot the error on log-scale against a log-epsilon that it corresponds very good to the line $-1 + 0.93 \times \log(\epsilon)$, which means that the error goes approximately as $C \times \epsilon^{0.93}$, slightly worse than a linear convergence in ϵ.

We now move on to consider the more interesting case of an NIG Lévy process and the approximation of such. We restrict our attention to the situation where we neglect the small jumps, and investigate the deviation between the "correct" portfolio strategy π^* and the approximative $\pi^*_{N,\epsilon}$.



(a) The exact optimal investment π^* (solid line) and the approximated one $\pi^*_{N,\epsilon}$ (dashed line).



(b) The error on a log-scale.

FIGURE 1. Error when $L(t)$ follows a Poisson process $N(t)$

Let the parameters be $\alpha = 50$, and $\delta = 0.03$ on a daily scale (this seems to be natural estimates of stocks, see Bølviken and Benth [7]). Furthermore we suppose $r = 0.04/250$ and $\xi = 0.02/250$. The interest rate is therefore 4% while logreturns have a mean of 2%, measured annually, when we assume there are 250 trading days in a year. The compound interest rate $\hat{\mu}$ on the average stock price becomes 0.093, or 9.3%, annually. This is clearly above r, implying that the condition $\hat{\mu} - r > 0$ is satisfied.

We computed the optimal π^* using $\epsilon = 10^{-10}$ to avoid the singularity at the center of the Lévy measure of the NIG. The resulting optimal investment strategy became $\pi^* = 70.59\%$. Next, by starting with $\epsilon = 0.0001$ and stepping down 0.00005, 0.00001, 0.000005, 0.000001,0.0000005, and 0.0000001, we get the errors as depicted in Figure 2. We remark that the truncation of the infinite integral limits goes at 1, where we observed that the tails of the NIG Lévy measure gave values which were of the magnitude 10^{-20}, where as in the center around 0 it reached values of the magnitude 10^{-2}. Strictly speaking, in this example we do not investigate the approximation of a

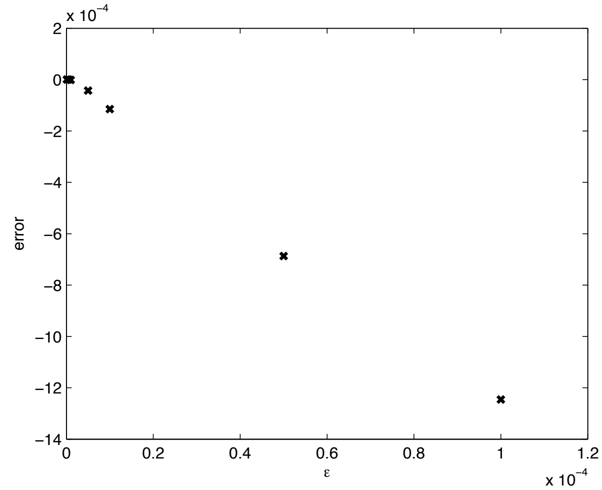


FIGURE 2. Error in the optimal control with a NIG Lévy process.

NIG Lévy process, but a Lévy process with *truncated* NIG Lévy measure at $|z| < 1$. Nevertheless, it reflects what to be expected for the true NIG case.

5. Convergence of the value functions

We have seen that the controls $\pi^*_{N,\epsilon}$ and $\pi^*_{W,\epsilon}$ converge to the control π^* as $\epsilon \to 0$. We next investigate the convergence of the corresponding value functions. We first discuss maximization of terminal wealth, and then move on to analyse the convergence of the optimal consumption and the value functions.

5.1. **Maximising expected utility of terminal wealth.**

We consider the problem of optimizing terminal wealth, which means there is no consumption involved in the control problem. We find the following for the case of truncation of the small jumps:

Proposition 3. *It holds for every $x \in \mathbb{R}_+$,*

$$\lim_{\epsilon \to 0} V_{N,\epsilon}(x) = V(x).$$

Proof. Recalling the wealth process $X^\pi(t)$ with no consumption, we find that it is a geometric jump diffusion process with constant coefficients and with solution

$$\begin{aligned}X^{\pi^*}(t) = x\exp\{&t(r + (\hat{\mu} - r)\pi) + t\int_{\mathbb{R}\backslash\{0\}} \ln(1 + \pi^*(e^z - 1)) - \pi^*(e^z - 1)1_{|z|<1}\nu(dz) \\ &+ \int_0^t \int_{\mathbb{R}\backslash\{0\}} \ln(1 + \pi^*(e^z - 1))\tilde{N}(ds, dz)\}.\end{aligned}$$

Applying the formula in Ex. 1.6 in Øksendal and Sulem [20] , we find

$$\begin{aligned}V(x) &= \mathbb{E}[U(X^{\pi^*}(T))] = \mathbb{E}\left[\frac{1}{\gamma}(X^{\pi^*}(T))^\gamma\right] \\ &= \frac{1}{\gamma}x^\gamma \exp\{\gamma T(r + (\hat{\mu} - r)\pi^*). \\ &\qquad + T\int_{\mathbb{R}\backslash\{0\}} (1 + \pi^*(e^z - 1))^\gamma - 1 - \gamma\pi^*(e^z - 1)1_{|z|<1}\nu(dz)\}\end{aligned}$$

If small jumps are neglected, we find similarly

$$\begin{aligned}
V_{N,\epsilon}(x) &= \mathbb{E}[U(X^*_{N,\epsilon}(T))] \\
&= \frac{1}{\gamma}x^\gamma \exp\{\gamma T(r + (\hat{\mu}_{N,\epsilon} - r)\pi^*_{N,\epsilon}) \\
&\qquad + T\int_{\mathbb{R}\setminus\{0\}} (1+\pi^*_{N,\epsilon}(e^z-1))^\gamma - 1 - \gamma\pi^*_{N,\epsilon}(e^z-1)1_{|z|<1}\nu_{N,\epsilon}(dz)\}. \qquad (29)
\end{aligned}$$

We have for $|z| < 1$

$$\left|(1+\pi^*_{N,\epsilon}(e^z-1))^\gamma - 1 - \gamma\pi^*_{N,\epsilon}(e^z-1)\right| \le \left|e^{\gamma z} - 1 - \gamma(e^z-1)\right|, \qquad (30)$$

and for $|z| > 1$,

$$\left|(1+\pi^*_{N,\epsilon}(e^z-1))^\gamma - 1\right| \le \left|e^{\gamma z} - 1\right|.$$

Both estimates are integrable on their definition area. As $\pi^*_{N,\epsilon}$ converges to π^* and $\hat{\mu}_{N,\epsilon}$ to $\hat{\mu}$, the proposition follows with Lebesgues convergence theorem. □

For the case of a Brownian approximation of the truncated small jumps we have:

Proposition 4. *It holds for every $x \in \mathbb{R}_+$,*

$$\lim_{\epsilon\to 0} V_{W,\epsilon}(x) = V(x).$$

Proof. A Brownian motion approximation of the small jumps as in (3.2) results in an additional Brownian component and $\sigma(\epsilon)$-terms in the wealth process:

$$\begin{aligned}
X^*_{W,\epsilon}(t) = x\exp\Big\{ & t\big[r + (\hat{\mu}_{W,\epsilon} - r)\pi_{W,\epsilon} - \frac{1}{2}\sigma^2(\epsilon)\pi^2_{W,\epsilon})\big] + \frac{1}{2}\sigma(\epsilon)\pi_{W,\epsilon}B_t \\
&+ t\int_{\mathbb{R}\setminus\{0\}} \ln(1+\pi^*_{W,\epsilon}(e^z-1)) - \pi^*_{W,\epsilon}(e^z-1)1_{|z|<1}\nu_{N,\epsilon}(dz) \\
&+ \int_0^t\int_{\mathbb{R}\setminus\{0\}} \ln(1+\pi^*_{W,\epsilon}(e^z-1))\tilde{N}_\epsilon(ds,dz)\}.
\end{aligned}$$

The value function becomes then

$$\begin{aligned}
V_{W,\epsilon}(x) &= \mathbb{E}[U(X^*_{W,\epsilon}(T))] \\
&= \frac{1}{\gamma}x^\gamma \exp\Big\{\gamma T(r + (\hat{\mu}_{W,\epsilon} - r)\pi^*_{W,\epsilon} - \frac{1}{2}\sigma^2(\epsilon)\pi^2_{W,\epsilon}) \\
&\qquad + T\int_{\mathbb{R}\setminus\{0\}} (1+\pi^*_{W,\epsilon}(e^z-1))^\gamma - 1 - \gamma\pi^*_{W,\epsilon}(e^z-1)1_{|z|<1}\nu_{N,\epsilon}(dz)\Big\}.
\end{aligned}$$

Hence, as for the case of $V_{N,\epsilon}$ above, $V_{W,\epsilon}(x)$ converges to $V(x)$. □

5.2. **Maximising exptected utility of consumption.** As we noted in Section 2, maximising wealth over optimal investment and consumption pairs (π, c) results in the same optimal strategy π^* as when maximising expected utility over terminal wealth. As our results show, approximations of these control problems leads to convergence of the optimal investment stategies, as well as the value functions for maximization of the utility of terminal wealth. We show next that including consumption does not alter these conclusions.

Proposition 5. *Let the value function be of the form (3) and suppose that the discount factor δ satisfies*

$$\delta > \gamma\hat{\mu} + \int_1^\infty e^{\gamma z} - 1\,\nu(dz).$$

Then it holds for every $x \in \mathbb{R}_+$,

$$\lim_{\epsilon\to 0} V_{N,\epsilon}(x) = V(x).$$

Proof. As the optimal consumption is a constant fraction of wealth, the wealth process is again a geometric jump diffusion process with constant coefficients and with solution

$$X^{\pi^*}(t) = x\exp\{t(r+(\hat{\mu}-r)\pi^*-c^*)+t\int_{\mathbb{R}\setminus\{0\}}\ln(1+\pi^*(e^z-1))-\pi^*(e^z-1)1_{|z|<1}\nu(dz)$$
$$+\int_0^t\int_{\mathbb{R}\setminus\{0\}}\ln(1+\pi^*(e^z-1))\tilde{N}(ds,dz)\}$$

where

$$c^*=\frac{1-\gamma}{\delta-k(\gamma)}.$$

Then the value function takes the form

$$\begin{aligned}V(x) &= \mathbb{E}[\int_0^\infty e^{-\delta t}\frac{(c^*(t))^\gamma}{\gamma}dt]\\ &= \frac{1}{\gamma}(c^*)^\gamma x^\gamma\int_0^\infty \exp\{t[-\delta+\gamma(r+(\hat{\mu}-r)\pi^*-c^*)\\ &\quad+\int_{\mathbb{R}\setminus\{0\}}(1+\pi^*(e^z-1))^\gamma-1-\gamma\pi^*(e^z-1)1_{|z|<1}\nu(dz)]\}dt\end{aligned}$$

where we have exchanged integration and expectation by appealing to Fubini's theorem and formula in Ex. 1.6 in Øksendal and Sulem [20]. For the value function to be finite, we must require that,

$$-\delta+\gamma(r+(\hat{\mu}-r)\pi^*-c^*)+\int_{\mathbb{R}\setminus\{0\}}(1+\pi^*(e^z-1))^\gamma-1-\gamma\pi^*(e^z-1)1_{|z|<1}\nu(dz)<0 \tag{31}$$

Condition (31) depends on π^* and c^*. With π^* bounded in $[0,1]$ and c^* positive we find

$$\gamma(r+(\hat{\mu}-r)\pi^*-c^*)\le\gamma\hat{\mu}.$$

Furthermore, the integrand in (31) is positive and increasing in π for $z>1$ and negative and decreasing in π for $z<1$. Then

$$\int_{\mathbb{R}\setminus\{0\}}(1+\pi^*(e^z-1))^\gamma-1-\gamma\pi^*(e^z-1)1_{|z|<1}\nu(dz)<\int_1^\infty e^{\gamma z}-1\nu(dz)$$

and condition (31) is fullfilled by the assumption on δ. Thus,

$$\begin{aligned}V(x) &= -\frac{1}{\gamma}(c^*)^\gamma x^\gamma[-\delta+\gamma(r+(\hat{\mu}-r)\pi^*-c^*)\\ &\quad+\int_{\mathbb{R}\setminus\{0\}}(1+\pi^*(e^z-1))^\gamma-1-\gamma\pi^*(e^z-1)1_{|z|<1}\nu(dz)]^{-1}.\end{aligned}$$

Neglecting small jumps, we find similarly

$$\begin{aligned}V_{N,\epsilon}(x) &= \frac{1}{\gamma}(c^*_{N,\epsilon})^\gamma\int_0^\infty x^\gamma\exp\{t[-\delta+\gamma(r+(\hat{\mu}_{N,\epsilon}-r)\pi^*_{N,\epsilon}-c^*_{N,\epsilon})\\ &\quad+\int_{\mathbb{R}\setminus\{0\}}(1+\pi^*_{N,\epsilon}(e^z-1))^\gamma-1-\gamma\pi^*_{N,\epsilon}(e^z-1)1_{|z|<1}\nu_{N,\epsilon}(dz)]\}dt\\ &= -\frac{1}{\gamma}(c^*_{N,\epsilon})^\gamma x^\gamma[-\delta+\gamma(r+(\hat{\mu}_{N,\epsilon}-r)\pi^*_{N,\epsilon}-c^*_{N,\epsilon})\\ &\quad+\int_{\mathbb{R}\setminus\{0\}}(1+\pi^*_{N,\epsilon}(e^z-1))^\gamma-1-\gamma\pi^*_{N,\epsilon}(e^z-1)1_{|z|<1}\nu_{N,\epsilon}(dz)]^{-1}\end{aligned}$$

With

$$c^*_{N,\epsilon}=\frac{1-\gamma}{\delta-k_{N,\epsilon}(\gamma)}$$

and

$$k_{N,\epsilon}(\gamma) = \gamma(r + (\hat{\mu}_{N,\epsilon} - r)\pi^*_{N,\epsilon}) + \int_{\mathbb{R}\backslash\{0\}} (1 + \pi^*_{N,\epsilon}(e^z - 1))^\gamma - 1 - \gamma\pi^*_{N,\epsilon}(e^z - 1)\nu_{N,\epsilon}(dz).$$

$k_{N,\epsilon}(\gamma)$ converges to $k(\gamma)$ as $\hat{\mu}_{N,\epsilon}$ and $\pi^*_{N,\epsilon}$ converges and by appealing to Lebesgue's convergence theorem using estimate (30). The integral

$$\int_{\mathbb{R}\backslash\{0\}} (1 + \pi^*_{N,\epsilon}(e^z - 1))^\gamma - 1 - \gamma\pi^*_{N,\epsilon}(e^z - 1)1_{|z|<1}\nu_{N,\epsilon}(dz)$$

did already appear in (29) in the value function connected to maximising terminal wealth, where its convergence was discussed. Then the proposition follows. □

Approximating the truncated jumps by a Brownian motion preserves convergence of the value function also in the consumption case, as the next result shows:

Proposition 6. *Let the value function be of the form (3). Then it holds for every $x \in \mathbb{R}_+$,*

$$\lim_{\epsilon \to 0} V_{W,\epsilon}(x) = V(x).$$

Proof. In this case the value function has the form

$$\begin{aligned} V_{W,\epsilon}(x) = \frac{1}{\gamma}(c^*_{W,\epsilon})^\gamma \int_0^\infty x^\gamma \exp\{t[-\delta + \gamma(r + (\hat{\mu}_{W,\epsilon} - r)\pi^*_{W,\epsilon} - \frac{1}{2}\sigma^2(\epsilon)(\pi^*_{W,\epsilon})^2 - c^*_{W,\epsilon}) \\ + \int_{\mathbb{R}\backslash\{0\}} (1 + \pi^*_{W,\epsilon}(e^z - 1))^\gamma - 1 - \gamma\pi^*_{W,\epsilon}(e^z - 1)1_{|z|<1}\nu_{N,\epsilon}(dz)]\}dt \end{aligned}$$

with

$$c^*_{W,\epsilon} = \frac{1-\gamma}{\delta - k_{W,\epsilon}(\gamma)}$$

A Brownian approximation as in (3.2) results in an additional $\sigma^2(\epsilon)$-term also in $k_{W,\epsilon}(\gamma)$:

$$\begin{aligned} k_{W,\epsilon}(\gamma) = \gamma(r + (\hat{\mu}_{W,\epsilon} - r)\pi^*_{W,\epsilon}) - \frac{1}{2}\sigma^2(\epsilon)(\pi^*_{W,\epsilon})^2\gamma(1-\gamma) \\ + \int_{\mathbb{R}\backslash\{0\}} (1 + \pi^*_{W,\epsilon}(e^z - 1))^\gamma - 1 - \gamma\pi^*_{W,\epsilon}(e^z - 1)\nu_{N,\epsilon}(dz). \end{aligned}$$

The convergence of the corresponding value function $V_{W,\epsilon}$ follows as for the case $V_{N,\epsilon}$ above. □

6. Convergence rate for the wealth process

Still remaining is the convergence of the wealth processes. Convergence in probability of the wealth processes is clear. For convergence in L_2 we can derive a rate which is, not surprisingly, proportional to $\sigma^2(\epsilon)$.

Proposition 7. *The wealth process $X_{N,\epsilon}(t)$ converges to the original process $X(t)$ in L_2. For every $T < \infty$ it is furthermore for the case of no consumption*

$$\sup_{t\in[0,T]} \mathbb{E}[|X(t) - X_{N,\epsilon}(t)|^2] \le K\sigma^2(\epsilon)$$

where K depends on T.

Proof. We can write the difference between the wealth processes as

$$
\begin{aligned}
X(t) - X_{N,\epsilon}(t) &= \int_0^t (r + (\hat{\mu} - r)\pi)X(s) - (r + (\hat{\mu}_{N,\epsilon} - r)\pi_{N,\epsilon})X_{N,\epsilon}(s)ds + \int_0^t cX(s) - c_{N,\epsilon}X_{N,\epsilon}(s)ds \\
&\quad + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \big(\pi X(s-) - \pi_{N,\epsilon}X_{N,\epsilon}(s-)1_{|z|>\epsilon}\big)(e^z - 1)\tilde{N}(ds, dz) \\
&= \int_0^t (r + (\hat{\mu} - r)\pi)(X(s) - X_{N,\epsilon}(s))ds \\
&\quad + \int_0^t X_{N,\epsilon}(s)\big((r + (\hat{\mu} - r)\pi) - (r + (\hat{\mu}_{N,\epsilon} - r)\pi_{N,\epsilon})\big)ds \\
&\quad - \int_0^t c\big(X(s) - X_{N,\epsilon}(s)\big)ds - \int_0^t (c - c_{N,\epsilon})X_{N,\epsilon}(s)ds \\
&\quad + \int_0^t \pi\big(X(s-) - X_{N,\epsilon}(s-)\big) \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)\tilde{N}(ds, dz) \\
&\quad + \int_0^t \int_{\mathbb{R}\setminus\{0\}} X_{N,\epsilon}(s-)(\pi - \pi_{N,\epsilon}1_{|z|>\epsilon})(e^z - 1)\tilde{N}(ds, dz)
\end{aligned}
$$

Then, it follows:

$$
\begin{aligned}
\mathbb{E}[|X(t) - X_{N,\epsilon}(t)|^2] \leq\ & c_1\mathbb{E}\Big[\Big(\int_0^t X(s) - X_{N,\epsilon}(s)ds\Big)^2\Big] \\
&+ c_2\mathbb{E}\Big[\Big(\int_0^t X_{N,\epsilon}(s)ds\Big)^2\Big]\Big\{\big((\hat{\mu} - r)\pi - (\hat{\mu}_{N,\epsilon} - r)\pi_{N,\epsilon}\big)^2\Big\} \\
&+ c_3\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}\setminus\{0\}} (X(s-) - X_{N,\epsilon}(s-))(e^z - 1)\tilde{N}(ds, dz)\Big)^2\Big] \\
&+ c_4\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}\setminus\{0\}} X_{N,\epsilon}(s-)(\pi - \pi_{N,\epsilon}1_{|z|>\epsilon})(e^z - 1)\tilde{N}(ds, dz)\Big)^2\Big]
\end{aligned}
$$

for constants $c_1, ..., c_4$. The constants denoted by c_1, c_2 are going to vary from step to step during this proof. For $t \leq T$ we find by Cauchy-Schwarz

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^t X(s) - X_{N,\epsilon}(s)ds\Big)^2\Big] &\leq T\int_0^t \mathbb{E}\Big[\big(X(s) - X_{N,\epsilon}(s)\big)^2\Big]ds \\
\mathbb{E}\Big[\Big(\int_0^t X_{N,\epsilon}(s)ds\Big)^2\Big] &\leq T\int_0^t \mathbb{E}\Big[X_{N,\epsilon}^2\Big]ds
\end{aligned}
$$

Furthermore, we find

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}\setminus\{0\}} (\pi(X(s-) - X_{N,\epsilon}(s-))(e^z - 1)\nu(dz)ds)\Big)^2\Big] \\
&\qquad = \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}\setminus\{0\}} \big(X(s-) - X_{N,\epsilon}(s-)\big)^2(e^z - 1)\nu(dz)ds\Big] \\
&\qquad = \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)^2\nu(dz)\int_0^t \mathbb{E}\Big[(X(s) - X_{N,\epsilon}(s))^2\Big]ds \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}\setminus\{0\}} X_{N,\epsilon}(s-)(\pi-\pi_{N,\epsilon})1_{|z|>\epsilon}(e^z-1)\tilde{N}(ds,dz)\Big)^2\Big] \\
&\quad = \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}\setminus\{0\}} X^2_{N,\epsilon}(s-)(\pi-\pi_{N,\epsilon}1_{|z|>\epsilon})^2(e^z-1)^2\nu(dz)ds\Big] \\
&\quad = \int_0^t \mathbb{E}\Big[X^2_{N,\epsilon}(s)\Big]ds \int_{\mathbb{R}\setminus\{0\}} (\pi-\pi_{N,\epsilon}1_{|z|>\epsilon})^2(e^z-1)^2)\nu(dz) \qquad (33)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}[|X(t)-X_{N,\epsilon}(t)|^2] \leq c_1 &\int_0^t \mathbb{E}\Big[\big(X(s)-X_{N,\epsilon}(s)\big)^2\Big]ds \\
&+ c_2 \int_0^t \mathbb{E}\big[X^2_{N,\epsilon}(s)\big]ds\Big\{\big((\hat{\mu}-r)\pi-(\hat{\mu}_{N,\epsilon}-r)\pi_{N,\epsilon}\big)^2 \\
&+ \int_{\mathbb{R}\setminus\{0\}} (\pi-\pi_{N,\epsilon}1_{|z|>\epsilon})^2(e^z-1)^2\nu(dz)\Big\}
\end{aligned}
$$

Also it follows that

$$
\begin{aligned}
\int_{\mathbb{R}\setminus\{0\}} (\pi-\pi_{N,\epsilon}1_{|z|>\epsilon})^2(e^z-1)^2\nu(dz) &= \int_{\mathbb{R}\setminus\{0\}} (\pi-\pi_{N,\epsilon}+\pi_{N,\epsilon}1_{|z|<\epsilon})^2(e^z-1)^2\nu(dz)\} \\
&\leq c_1 \int_{\mathbb{R}\setminus\{0\}} (e^z-1)^2\nu(dz)(\pi-\pi_{N,\epsilon})^2 + c_2\int_{|z|<\epsilon}(e^z-1)^2\nu(dz) \\
&\leq c_1|\pi-\pi_{N,\epsilon}|^2 + c_2\sigma^2(\epsilon) \qquad (34)
\end{aligned}
$$

and

$$
\begin{aligned}
\big((\hat{\mu}-r)\pi-(\hat{\mu}_{N,\epsilon}-r)\pi_{N,\epsilon}\big)^2 &= \big((\hat{\mu}-r)(\pi-\pi_{N,\epsilon})+\pi_{N,\epsilon}(\hat{\mu}-\hat{\mu}_{N,\epsilon})\big)^2 \\
&\leq c_1|\pi-\pi_{N,\epsilon}|^2 + c_2|\hat{\mu}-\hat{\mu}_{N,\epsilon}|^2. \qquad (35)
\end{aligned}
$$

Therefore,

$$
\mathbb{E}[|X(t)-X_{N,\epsilon}(t)|^2] \leq \tilde{c}_1 \int_0^t \mathbb{E}[\big(X(s)-X_{N,\epsilon}(s)\big)^2]ds + \tilde{c}_2\big((|\pi-\pi_{N,\epsilon}|^2+\sigma^2(\epsilon)\big)\int_0^t \mathbb{E}[X^2_{N,\epsilon}(s)]ds.
$$

Then it follows by Gronwall's inequality:

$$
\begin{aligned}
\mathbb{E}[|X(t)-X_{N,\epsilon}(t)|^2] &\leq \tilde{c}_2 \int_0^t e^{\tilde{c}_1(t-s)}\mathbb{E}[X^2_{N\epsilon}]ds\Big(\sigma^2(\epsilon)+(\pi-\pi_{N\epsilon})^2\Big) \\
&\leq c_1\sigma^2(\epsilon)+c_2\sigma^4(\epsilon) \\
&\leq c_1\sigma^2(\epsilon)
\end{aligned}
$$

Hence, for every $T<\infty$ we conclude

$$
\sup_{t\in[0,T]} \mathbb{E}[|X(t)-X_{N,\epsilon}(t)|^2] \leq K\sigma^2(\epsilon)
$$

where K depends on T, and the Proposition follows.

□

The convergence and convergence rates in this paper are analysed for the specific case of power utility in a Merton framework. The proofs, especially for the convergence rate of the optimal control, depend on features of the concrete form of the solutions in this specific setting. In a more general setting a concrete solution is not available. Additionally it is not clear if the optimal control and the consumption rate are constant in time.

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